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J. Phys. A: Math. Gen., Vol. 10, No. 1, 1977. Printed in Great Britain. © 1977

Quantum field theory on a cone

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Received 24 June 1976

Abstract. The expressions derived by Sommerfeld and Carslaw for the Green functions and diffusion kernels in a wedge of arbitrary angle are shown to be useful in discussions of the Feynman Green function in Rindler space and other space–time metrics.

1. Introduction

The fact that the polar angle ϕ, on a plane say, is not a single-valued function of position leads to a number of formal difficulties. For example in quantum mechanics the definition of an angle operator $\hat{\phi}$ in a Hilbert space is impossible directly. Also in functional integrals it is most convenient to have all the variables running from $-\infty$ to $+\infty$ in order to apply Gaussian integrals.

Of course on the plane it is always possible to use Cartesian coordinates so that one need not trouble about such questions. However, we ought to be able to use whatever coordinate system we like and this includes polar coordinates; and there may be occasions when angular coordinates are obligatory.

These problems are well known and no doubt everyone who has thought about them will have their own approach. What I wish to give in this paper are some personal comments which others might like to consider. Mainly I wish to resurrect some old and apparently forgotten work of Sommerfeld (1897, see Frank and von Mises 1935, chap. 20) and elaborated by others (e.g. Carslaw 1898, 1910, 1919).

Sommerfeld's general idea of using a Riemann surface has not been lost, of course, but I wish to draw attention to the precise analytical expressions derived by these authors as being particularly relevant for some topics of present day physics.

The reason for this is that the polar coordinate form of the metric,

$$dr^2 + r^2\, d\phi^2 \tag{1}$$

occurs in a number of places. Of most interest for us is the fact that (1) is the Euclidean form of the 'Rindler metric' of two-dimensional Minkowski space–time,

$$ds^2 = z^2\, dv^2 - dz^2 \tag{2}$$

obtained by setting the 'time' v equal to $i\phi$ and z equal to r.

If we are going to use this continuation we do not necessarily want to identify ϕ and $\phi + 2\pi$. Rather we would like to make the periodicity arbitrary. If ϕ and $\phi + \beta$ are identified then (1) describes the geometry on a cone of semi-angle $\sin^{-1}(\beta/2\pi)$, hence the title of this paper.

2. The quantum mechanical propagator

We are particularly interested in quantum mechanics and quantum field theory on the cone. Thus we ask for the propagator $K_\beta(r, \phi, r', \phi', \tau)$ which satisfies Schrödinger's equation

$$\left(\mathrm{i}\frac{\partial}{\partial\tau}+\nabla^2\right)K_\beta(r, \phi, r', \phi', \tau)=\mathrm{i}\delta(\tau)\delta(\phi-\phi')\frac{\delta(r-r')}{(rr')^{1/2}} \tag{3}$$

for $|\phi|$ and $|\phi'|<\beta/2$, and has period β in the angle variables.

If we set $\tau\to-\mathrm{i}\tau$ we are considering diffusion and all we have to do is to read off the solution from Carslaw's paper (1909). I give the result for $\beta=\infty$ first, both in 'series' form,

$$K_\infty(r, \phi, r', \phi', \tau)=-\frac{\mathrm{i}}{4\pi\tau}\,\mathrm{e}^{\mathrm{i}(r^2+r'^2)/4\tau}\int_{-\infty}^{\infty}\mathrm{d}\mu\;\mathrm{e}^{\mathrm{i}\mu(\phi-\phi')}\,\mathrm{e}^{-\frac{1}{2}\mathrm{i}|\mu|\pi}\,J_{|\mu|}\left(\frac{rr'}{2\tau}\right) \tag{4}$$

and as a contour integral,

$$K_\infty(\boldsymbol{r}, \boldsymbol{r}', \tau)=-\frac{1}{8\pi^2\tau}\int_{\mathrm{A}}\mathrm{e}^{\mathrm{i}(\boldsymbol{\rho}-\boldsymbol{r}')^2/4\tau}\frac{\mathrm{d}\alpha}{\alpha-\phi}, \qquad (\tau>0) \tag{5}$$

with $\boldsymbol{r}=(r, \phi)$, $\boldsymbol{r}'=(r', \phi')$ and $\boldsymbol{\rho}=(r, \alpha)$. The contour A has two branches, one in the upper half α plane from $(\phi'+\pi-\epsilon)+\mathrm{i}\infty$ to $(\phi'-\pi-\epsilon)+\mathrm{i}\infty$ and the other in the lower half-plane from $(\phi'-\pi+\epsilon)-\mathrm{i}\infty$ to $(\phi'+\pi+\epsilon)-\mathrm{i}\infty$ (cf Carslaw 1919, figure 1).

K_∞ is the propagator on Sommerfeld's infinitely-sheeted Riemann surface (cf Franck and von Mises 1935, pp 821, 839, 814). To obtain the propagator on the cone, one simply performs the periodicity sum

$$K_\beta(\boldsymbol{r}, \boldsymbol{r}', \tau)=\sum_{m=-\infty}^{\infty}K_\infty(\boldsymbol{r}_m, \boldsymbol{r}', \tau) \tag{6}$$

where $\boldsymbol{r}_m=(r, \phi+m\beta)$, which gives a quantity of period β (note that our β is twice Carslaw's).

The sum turns the Fourier integral in (4) into a Fourier series in the usual way,

$$K_\beta(\boldsymbol{r}, \boldsymbol{r}', \tau)=-\frac{\mathrm{i}}{2\beta\tau}\,\mathrm{e}^{\mathrm{i}(r^2+r'^2)/4\tau}\sum_{n=-\infty}^{\infty}\mathrm{e}^{2\pi\mathrm{i}n(\phi-\phi')/\beta}\,\mathrm{e}^{-\mathrm{i}|n|\pi^2/\beta}\,J_{2|n|\pi/\beta}\left(\frac{rr'}{2\tau}\right). \tag{7}$$

While the contour form (5) yields

$$K_\beta(\boldsymbol{r}, \boldsymbol{r}', \tau)=-\frac{\mathrm{i}}{4\pi\beta\tau}\int_{\mathrm{A}}\mathrm{e}^{\mathrm{i}(\boldsymbol{\rho}-\boldsymbol{r}')^2/4\tau}\frac{\mathrm{e}^{2\pi\mathrm{i}\alpha/\beta}}{\mathrm{e}^{2\pi\mathrm{i}\alpha/\beta}-\mathrm{e}^{2\pi\mathrm{i}\phi/\beta}}\,\mathrm{d}\alpha, \tag{8}$$

which is easily shown to be identical to (7).

As a minor point, note that a cotangent form could be substituted for the ratio of exponentials in (8) (cf Bromwich 1915).

When $\beta=2\pi$ these expressions reduce to the standard propagator

$$-\frac{\mathrm{i}}{4\pi\tau}\,\mathrm{e}^{\mathrm{i}(\boldsymbol{r}-\boldsymbol{r}')^2/4\tau}. \tag{9}$$

To obtain this from (8) the contour A is deformed into a loop around $\alpha=\phi\,(|\phi|<\pi)$ and two vertical lines oppositely directed and a distance 2π apart. These lines give equal and opposite contributions if the integrand has period 2π, which it does if $\beta=2\pi$, and we are left with (9) as the simple pole contribution at $\alpha = \phi$. (This is actually the reverse of the arguments used by Sommerfeld and Carslaw to motivate the form (8).)

The line contributions also cancel if β is an integral fraction of 2π, $\beta = 2\pi/p$ and we then get just the poles at $\alpha = \phi + 2s\pi/p$, $s = 0, \pm1, \pm2, \ldots$ subject to $\alpha - \phi' \leq \pi$ (cf Bromwich 1915). This is the case in which the simple image device works for finding the propagator or Green function in a wedge of angle $\beta/2$ with Dirichlet or Neumann boundary conditions.

When β is arbitrary this deformation of the contour in (8) will yield expression (9) plus a remainder which consists of the contributions of any other poles (there are none if $\beta > 2\pi$) and of the infinite lines. This remainder can be thought of as a non-perturbative contribution to K_β. The reason for saying this is that the cone is locally flat, except at the apex, so that all the coefficients in the expansion of $K_\beta(\tau)$ in powers of τ (the 'perturbation' expansion) are zero, apart from the first one.

Because the cone is flat almost everywhere it might be imagined that the propagator should be represented as a 'sum over classical paths' with (9) as the leading (i.e. direct path) contribution. I would rather look upon equation (6), with (5), in this light. Thus expression (5) is the (exact) semi-classical propagator on the flat space $\mathcal{M}_\infty$ and the periodicity sum (6) is a sum over classical paths on $\mathcal{M}_\infty$ and gives the propagator on $\mathcal{M}_\beta = \mathcal{M}_\infty/\mathbb{Z}_\infty(\beta)$ where $\mathbb{Z}_\infty(\beta)$ is the infinite cyclic group with period β, and the division identifies points $(r, \phi+\beta)$ and (r, ϕ) on $\mathcal{M}_\infty$.

The manifold $\mathcal{M}_\infty$ is simply connected while $\mathcal{M}_\beta$ has, in general, the fundamental group $\mathbb{Z}_\infty$ if we do not allow paths to go through the apex of the cone, a singular point.

The periodicity sum (6) is now just the expression for the propagator, K_β, on the multiply-connected space in terms of that, K_∞, on the universal covering space (Laidlaw and Morette-De Witt 1971, Schulman 1971, Dowker 1972).

Strictly speaking this interpretation is incorrect for the special and isolated case when $\beta = 2\pi$ for then $\mathcal{M}_\beta$ is the plane and is simply connected. However, since this exception is a set of measure zero we can let the interpretation stand for reasons of economy and continuity.

In general one is allowed to have phase factors multiplying each term in the sum (6) (see the last references). Let us see what this freedom produces. Define the propagator $K_{\beta,\delta}$ by

$$K_{\beta,\delta}(\boldsymbol{r}, \boldsymbol{r}', \tau) = \sum_{m=-\infty}^{\infty} \mathrm{e}^{2\pi\mathrm{i}m\delta} K_\infty(\boldsymbol{r}_m, \boldsymbol{r}', \tau). \tag{10}$$

Then the equations corresponding to (7) and (8) are

$$K_{\beta,\delta} = -\frac{\mathrm{i}}{2\beta\tau}\, \mathrm{e}^{\mathrm{i}(r^2+r'^2)/4\tau} \sum_{-\infty}^{\infty} \mathrm{e}^{2\pi\mathrm{i}(n+\delta)(\phi-\phi')/\beta}\, \mathrm{e}^{-\mathrm{i}|n+\delta|\pi^2/\beta}\, \mathrm{J}_{2\pi|n+\delta|/\beta}\left(\frac{rr'}{2\tau}\right) \tag{11}$$

and

$$K_{\beta,\delta} = -\frac{1}{8\pi\tau\beta} \int_{\mathrm{A}} \mathrm{e}^{\mathrm{i}(\boldsymbol{\rho}-\boldsymbol{r}')^2/4\tau} \frac{\mathrm{e}^{\mathrm{i}\pi(\alpha-\phi)(2\delta-1)/\beta}}{\sin[\pi(\alpha-\phi)/\beta]}\, \mathrm{d}\alpha, \qquad (0<\delta\leq 1). \tag{12}$$

When $\beta = 2\pi$ expression (11) is, as expected, that which emerges from an exact solution for the Aharonov–Bohm set up (Aharonov and Bohm 1959, especially § 4). There, the parameter δ is the electromagnetic flux through the axis (cf the discussion in Schulman 1971, 1975). In general δ will be related to the physical situation 'inside' the origin, $r = 0$. If there is no physics inside the origin, other than that producing the conical singularity, δ will be zero.

3. Green functions

In the works of Sommerfeld and Carslaw the cone, or wedge, was a real one and interest centred on diffusion in it or on the scattering of electromagnetic waves by it. My concern is not with these problems although one can quite easily set up quantum field theory in such a spatial wedge. It is straightforward to modify the results of Carslaw (1919) and MacDonald (1902, 1915) so as to give the Feynman Green function in the wedge, and related geometries. (This Green function also follows from the propagator $K_\beta(\tau)$ by treating τ as a proper time.) It is then possible to calculate the vacuum average of the stress–energy tensor for example (Dowker, unpublished).

I wish to dwell on the idea that the wedge-like geometry is the Euclidean form of a space–time geometry. This was briefly mentioned in § 1, where the 'real' and 'Rindler' wedges were related by continuation. Interest centres on the Feynman Green function in the Rindler wedge. This can be obtained by continuation of the Green function in the Euclidean region which itself can be expressed as an integral over the diffusion kernel $K_\beta(-\mathrm{i}\tau)$.

The details of this continuation will be given since everything can be followed through explicitly.

We use Rindler coordinates z and v rather than r and ϕ, and define the Green function $G_\beta(\zeta)$ by

$$G_\beta(\zeta)=\frac{\zeta}{4\pi\beta}\int_C\frac{\mathrm{d}\tau}{\tau}\,\mathrm{e}^{-\kappa^2\tau}\int_A \mathrm{e}^{-(z^2+z'^2-2zz'\cos\alpha')/4\tau}\frac{\mathrm{e}^{2\pi\mathrm{i}\alpha'/\beta}}{\mathrm{e}^{2\pi\mathrm{i}\alpha'/\beta}-\mathrm{e}^{2\pi\mathrm{i}\zeta(v-v')/\beta}}\,\mathrm{d}\alpha', \tag{13}$$

where $v-v'$ is real and ζ is a complex continuation variable (our ζ is i times that of Schwinger 1959). The contour C begins at the origin and ends at complex infinity. As ζ is changed we may have to deform C and A to give an analytical continuation of $G_\beta(\zeta)$ from the diffusion start, $\zeta=1$, to the quantum mechanics finish, $\zeta=\mathrm{i}$. We could consider C and A to be dependent on ζ.

There are two problems of convergence, one at $\tau=\infty$ and the other at $\tau=0$. The latter is the most awkward requirement. The contour A must always run in regions of the α' plane that make the integrand converge at $\tau=0$. As ζ approaches i this severely constricts the contour C.

Firstly consider the case $\beta=\infty$ and enquire after the analytical structure of $G_\infty(\zeta)$. The integrand has a pole at $\alpha'=\zeta(v-v')$. If we choose C to be the positive real τ axis the contour A must run in the shaded area of figure 1 for convergence at $\tau=0$. As ζ varies $G_\infty(\zeta)$ will be analytic at least until the pole at $\zeta(v-v')$ hits A, in which case the function $G_\infty(\zeta)$ must be continued, if possible, by adjusting the contour C. This shifts the shaded region and allows the A contour to be deformed away from the pole.

When C is the positive real τ axis, the indicated A contour in figure 1 is the optimum one and leads to the corresponding $G_\infty(\zeta)$ being analytic in the shaded region of figure 2.

As $|v-v'|$ tends to zero, for $\alpha_1=0$, this region turns into the (smaller) double wedge area, $\xi^2-\eta^2>0$, $\zeta=\xi+\mathrm{i}\eta$. This behaviour is expected since $\alpha_1=0$ means that the time-like contribution dominates the space–time interval. The gap on the imaginary axis in figure 2 is due to effect of the space-like contribution to the interval.

If it is desired to reach a point that lies in the unshaded part of figure 2, i.e. if a continuation to a time-like separation is wanted, the contour C can be altered. Bearing in mind that we want to continue from $\zeta=1$ to $\zeta=\mathrm{i}$ the C contour is rotated from the

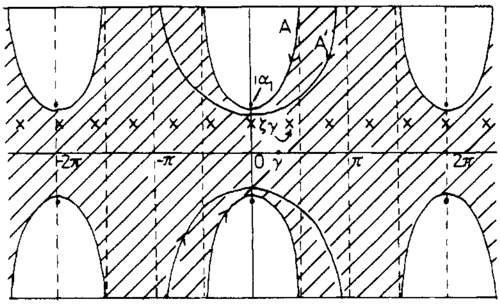

Figure 1. Region of α' plane in which contour A must run for convergence at $\tau = 0$ if contour C is along positive real τ axis. The contour A′ corresponds to that in equation (14). It *can* run in the unshaded part but must pass below the branch point $i\alpha_1 = i\cosh^{-1}(z^2 + z'^2/2zz')$ and need not approach $\pm i\infty$ in the shaded region. The crosses indicate a typical series of poles $\alpha' = \zeta\gamma + m\beta$, $(\gamma \equiv v - v')$.

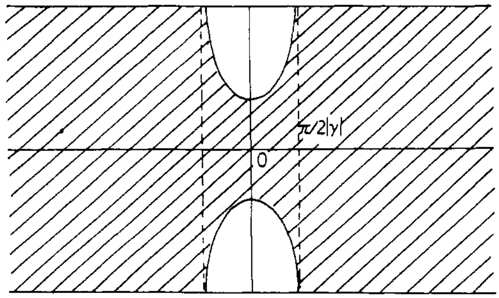

Figure 2. Domain of definition of the function $G_\infty(\zeta)$ in ζ plane for A contour as in figure 1.

positive real axis almost to the positive imaginary one. In this limiting case the A contour must run in the shaded region in figure 3. Again, the contour indicated is the optimum one and gives the shaded area in figure 4 as the points at which the $G_\infty(\zeta)$ is now defined. As $|v - v'|$ tends to zero this region becomes the two quadrants, $\xi\eta > 0$, as expected.

If the τ contour is rotated almost to the negative imaginary axis the pattern on the top of figure 3 is shifted by π to the right while that on the bottom is moved by the same amount to the left, and similarly for figure 4. This allows a continuation from $\zeta = 1$ to $\zeta = -i$.

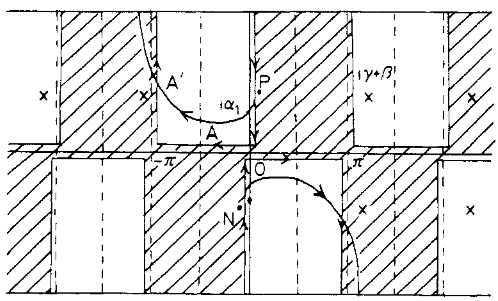

Figure 3. Same as figure 1 except that C has been rotated to almost the positive imaginary τ axis, and the contour A′ refers to that in equation (15). The poles in the fourth quadrant are the reflections in the origin of those in the second and we have set ζ equal to i.

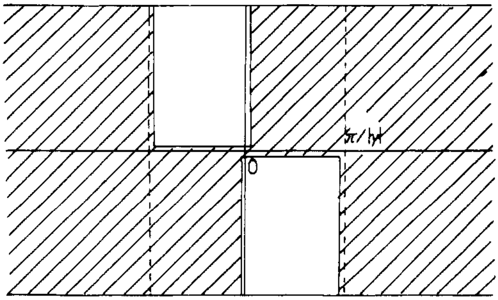

Figure 4. Domain of definition of $G_\infty(\zeta)$ for A as in figure 3.

All this shows that we can extend the definition of the function $G_\infty(\zeta)$ to the entire ζ plane with the exception of the sections of the imaginary axis, $\pm i\infty$ to $\pm[\cosh^{-1}(z'^2+z^2/2zz')]/|v-v'|$, which can be considered to form the edges of a cut.

The function of z, z' v, v' that we get by evaluating $G_\infty(\zeta)$ at $\zeta=\mathrm{i}$ is the Feynman Green function (e.g. Schwinger 1959). On figure 3 this corresponds to the points P and N, for positive and negative time differences, $v-v'$, respectively.

The τ integration in (13) can be performed to give the more explicit forms

$$G_\infty(\zeta)=\frac{\mathrm{i}\zeta}{4\pi^2}\int_{\mathrm{A}'} K_0[\kappa(z^2+z'^2-2zz'\cos\alpha')^{1/2}]\frac{\mathrm{d}\alpha'}{\alpha'-\zeta(v-v')} \tag{14}$$

and

$$G_\infty(\zeta)=-\frac{\zeta}{8\pi}\int_{\mathrm{A}'} H_0^{(2)}[\kappa(2zz'\cos\alpha'-z^2-z'^2)^{1/2}]\frac{\mathrm{d}\alpha'}{\alpha'-\zeta(v-v')} \tag{15}$$

for the C and A′ contours corresponding to figures 1 and 3 respectively.

These expressions are closely related to the results of Carslaw (1919, especially § 5) and MacDonald (1902, 1915).

The case of a general β now presents itself. The integrand of (13) has poles at $\alpha'=\zeta(v-v')+m\beta$, $m=0,1,2\ldots$, indicated typically by crosses in figure 1.

The domain of definition in the ζ plane is now determined by requiring that as ζ varies none of these poles hits the A contour. The resulting region can be obtained from that of figure 2 by excluding not only the unshaded part but also all its translates by $m\beta/|v-v'|$. If β is smallish this produces a strip with scalloped edges between roughly $\pm\alpha_1/|v-v'|$ where α_1 is the beginning of the cut on the imaginary α' axis.

For the limiting case of figure 4, which is needed for a continuation to $\zeta=\mathrm{i}$, the domain of definition is obtained by translating the unshaded rectangular portions. If $\beta<\pi$ this leaves only a thin strip around the real axis. In other words, as ζ varies from say 1 to i those poles lying originally between $v-v'$ and $v-v'-\pi$ (for convenience we take $v-v'>0$) cannot avoid the upper A contour before $\zeta=\mathrm{i}$ is reached.

The conclusion is that for $\beta<\pi$ this continuation to the Minkowski signature is not possible. It is allowed however if $\beta>\pi$ in which case we find

$$\begin{aligned} G_\beta(\mathrm{i})&=\frac{1}{4\beta}\int_{\mathrm{A}'} H_0^{(2)}[\kappa(2zz'\cos\alpha'-z^2-z'^2)^{1/2}]\frac{\mathrm{e}^{2\pi\mathrm{i}\alpha'/\beta}}{\mathrm{e}^{2\pi\mathrm{i}\alpha'/\beta}-\mathrm{e}^{-2\pi(v-v')/\beta}}\,\mathrm{d}\alpha' \\ &=\sum_{m=-\infty}^{\infty} G_\infty(z,z',v,v'+\mathrm{i}m\beta), \end{aligned} \tag{16}$$

where G_∞ is given by (15) with $\zeta=\mathrm{i}$.

Precisely what the significance of the value $\beta = \pi$ is I am not certain and for the remainder of this paper I shall choose $\beta > \pi$. (But see the final paragraph of § 5.)

Equation (16) corresponds to the pre-image sum (6) and the remarks concerning classical paths etc can also be applied to (16).

We may note that for $\beta = 2\pi$, (16) yields the standard two-dimensional Feynman Green function

$$G_F(x, x') = G_{2\pi}(i) = \tfrac{1}{4}H_0^{(2)}[\kappa((x-x')^2 - i\epsilon)^{1/2}]. \tag{17}$$

Before discussing the significance of the result (16) I wish to re-arrange it slightly. The typical distribution of the poles of the integrand of (16) is indicated by the upper series of crosses in figure 3 for a positive time difference $v - v'$. Separating off the pole at P (corresponding to $m = 0$), those other poles to the right (left) of the imaginary axis are given by positive (negative) m and correspond precisely to the sum (16).

This sum is now arranged to run over just positive m by setting $\alpha' \to -\alpha'$ in the $m < 0$ terms and using the symmetry of the integrand and contour A′.

Geometrically the poles in the second quadrant have been reflected in the origin to give poles in the fourth quadrant at the complex conjugate points to those in the first one.

An identical configuration results if $v - v'$ is negative so that this decomposition can be written in general

$$G_\beta(v) = G_\infty(v) + \sum_{m=1}^{\infty} [G_\infty^{(+)}(v - im\beta) - G_\infty^{(-)}(v + im\beta)] \tag{18}$$

where $G^{(\pm)}$ are the positive and negative frequency parts of the Pauli–Jordan function, i.e.

$$G_\infty(v) = \theta(v)G_\infty^{(+)}(v) - \theta(-v)G_\infty^{(-)}(v).$$

The elementary rules for extending the time variable in Green functions into the complex plane have been discussed by Schwinger (1958, 1959) and Nakano (1959).

4. Finite-temperature Green functions: Fock space

Equation (16) is identical to the result derived by Dowker and Critchley (1976a) for a finite-temperature Green function in a static space–time. (It should be said that this general result, for flat space, was briefly noted some time ago by Symanzik (1966).)

In our units the constant temperature T_0 is determined by the period β in imaginary time by $kT_0 = \beta^{-1}$.

The 'particles' that are in thermal equilibrium at the temperature T_0 are those associated with the Fock space, the vacuum of which determines the zero-temperature Green function G_∞ on the infinitely-sheeted manifold $\mathcal{M}_\infty$ in the usual way:

$$G_\infty(x, x') = i\frac{{}_\infty\langle 0_{\text{out}}|T\{\hat{\phi}(x)\hat{\phi}^\dagger(x')\}|0_{\text{in}}\rangle_\infty}{{}_\infty\langle 0_{\text{out}}|0_{\text{in}}\rangle_\infty}. \tag{19}$$

This Green function is the 'Rindler' Green function and corresponds to the mode decomposition discussed by Fulling (1973, § IIB, see also Unruh 1976). An explicit eigenfunction evaluation of this Green function has been carried out by Candelas and Raine (1976) (cf Carslaw and Jaeger 1959, § 14.14). The associated particles are called 'Rindler' particles.

The local temperature T is defined to be that measured by an observer situated at a fixed spatial point, defined by fixed values of the spatial coordinates. It is related to T_0 by the Tolman relation $T = T_0(g_{00})^{-1/2}$, (see e.g. Landau and Lifshitz 1958, § 27, Balazs 1958).

Physically in the present case where we identify z and v with Rindler coordinates an observer at constant z is moving with acceleration z^{-1} with respect to the Minkowski inertial frame (Rindler 1966) and so such an observer would measure a local temperature of $T = (\text{acceleration})/k\beta$.

In the special case when $\beta = 2\pi$ (i.e. no conical singularity), for which G_β is the conventional Feynman Green function (17), this result is just that of Davies (1975) but derived without the need for a reflecting wall. Unruh (1976) has also discussed this 'phenomenon'.

These authors use mode decompositions and it is clear that the present method is a means of bypassing these expressions and the attendant Bogoliubov transformations (see Fulling 1973, § IIC). In fact the contour integral in (16) can be thought of as the Green function analogue of the Bogoliubov transformation.

Unruh (1976) and Gibbons and Hawking (1975) would say that this result proves that an accelerated observer in flat space–time would see a thermal flux of particles with temperature T. The argument involves a *gedanken* construction with a particle detector but I do not wish to investigate this side of the discussion since I am more interested in the mathematical properties of the Green functions etc.

Lest it be thought that our work is relevant for only the flat-space case I turn now to the Schwarzschild solution, already discussed by Gibbons and Perry (1976) from the finite-temperature Green function point of view. Some comments were also given in our earlier work (Dowker and Critchley 1976a).

There are various ways of writing the Schwarzschild solution. I choose the following modification of the static form:

$$-\mathrm{d}s^2 = \frac{32M^3}{r}\,\mathrm{e}^{-r/2M}\left[\mathrm{d}z^2 - z^2\mathrm{d}\left(\frac{t}{4M}\right)^2\right] + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) \tag{20}$$

because it incorporates a Rindler-like part. The variable t is the static time and z is related to r by

$$z = \exp(r^*/4M) = (r/2M - 1)^{1/2}\exp(r/4M).$$

A conical structure has now been exhibited in the Schwarzschild metric and the results of the previous sections are easily applied if we identify $t/4M$ with v. We shall not be able to give explicit expressions for the Green functions, because of the complicated radial equation, but the general structure of equation (16) will remain.

Thus G_β can be expressed in terms of $G_{2\pi}$ over a contour A′,

$$G_\beta(x, x') = \int_{\mathrm{A}'} G_{2\pi}(2zz'\cos\alpha' - z^2 - z'^2, \ldots)\frac{\beta^{-1}\,\mathrm{e}^{2\pi\mathrm{i}\alpha'/\beta}}{\mathrm{e}^{2\pi\mathrm{i}\alpha'/\beta} - \mathrm{e}^{-\pi(t-t')/2M\beta}}\,\mathrm{d}\alpha' \tag{21}$$

because $G_{2\pi}(x, x')$ is a function of $t - t'$ through only the combination $z^2 + z'^2 - 2zz'\cosh(t - t'/4M)$, as in the flat case.

The periodic sum form of (21) is, as before,

$$G_\beta(t - t') = \sum_{m=-\infty}^{\infty} G_\infty(t - t' - \mathrm{i}m4M\beta). \tag{22}$$

G_∞ is the Feynman Green function on the manifold which is the continuation of Sommerfeld's infinitely-sheeted space $\mathcal{M}_\infty$, extended by the extra variables θ and ϕ, while G_β is the function on the extended continuation of the multiply-connected space $\mathcal{M}_\beta = \mathcal{M}_\infty/\mathbb{Z}_\infty(\beta)$.

If $\beta = 2\pi$ (no conical singularity) $G_{2\pi}$ will be the 'correct' Feynman Green function on the Schwarzschild manifold and if interpreted as a finite-temperature Green function according to (22), will give the Hawking temperature $T_0 = (8\pi kM)^{-1}$. G_∞ is then to be thought of as the zero-temperature Green function and would result from an expression like (19) using the 'naive' static mode decomposition discussed by Unruh (1976, see also Fulling 1973).

Exactly the same considerations apply to de Sitter space. The static form of the metric can be rewritten so as to display a Rindler-like part,

$$ds^2 = \frac{4}{a^2}(1+z^2)^{-2}[z^2\,\mathrm{d}(at)^2 - \mathrm{d}z^2] - \frac{1}{a^2}\left(\frac{1-z^2}{1+z^2}\right)^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)$$

with

$$z^2 = \frac{1-ar}{1+ar},$$

and so expressions virtually identical to (21) and (22) can be derived, the only difference being that $(4M)^{-1}$ is replaced by a, the radius of the S_4^1 hypersphere.

In this case the Green function $G_{2\pi}$ is that one calculated by a number of authors (e.g. Tagirov 1973, Candelas and Raine 1975, Dowker and Critchley 1976b) by a variety of methods and, precisely as above, will yield a temperature of $T_0 = a/2\pi k$ if interpreted as a finite-temperature Green function.

5. Discussion and conclusion

The purpose of this paper was to indicate the relevance of some old work of Sommerfeld, and I feel that this has been accomplished. The relations between the various Green functions have been expressed as contour integrals and Carslaw's periodicity sum has been equated to an expression for the finite-temperature Green function derived in an earlier paper. In particular the zero-temperature Green function is the function on a simply-connected infinitely-sheeted 'Riemann surface'. The corresponding vacuum $|0\rangle_\infty$ is what Unruh (1976) would call the 'η-vacuum' while that one associated with $G_{2\pi}$ (the 'correct' Green function) is the 'ξ-vacuum'. In the present approach the boundary conditions are built in.

I should mention here that Fulling (1973) has given a discussion of the Fock spaces associated with manifolds that are multiply connected *spatially* (boxes with periodic boundary conditions).

Only the mathematical expressions have been given here. I reserve applications and extensions for another time.

Finally a technical 'question mark'. When (15) was continued in ζ a difficulty arose for $\beta < \pi$ because the A contour was restricted to lie in the shaded parts of figures 1 and 3. However, the τ integral can be done, for any β, if C is along the real positive τ axis and ζ lies in the scalloped region. This yields an integrand involving K_0 and now, apparently, the restrictions on the A contour can be relaxed allowing the continuation in ζ with no problems. This yields equation (16) for any β.

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